Indian Statistical Institute, Bangalore
B. Math.
Third Year, Second Semester
Analysis IV

Mid Term Examination | Date : 07 March 2022
Maximum marks: 30 | Time: 90 minutes
Instructor: C R E Raja

Answer any five and all questions carry equal marks

1. Let $\mathcal{B}$ be a closed subalgebra of all real-valued continuous functions on a compact metric space X. Then $|f| \in \mathcal{B}$ for any $f \in \mathcal{B}$.

2. Let X be a compact metric space and $E \subset C(X)$ such that E is equicontinuous and point-wise bounded. Then prove that every sequence in E has a convergent subsequence.

3. Let $f_n\colon [0,1] \to \mathbb{R}$ be a sequence of differentiable functions such that $|f_n'(x)| \leq (1+1/n)$. Prove that (f_n) has a convergenct subsequence if $(f_n(0))$ is a bounded sequence.

4. Determine optimal conditions under which a set of polynomials is compact in the space of continuous functions on $[0,1]$.

5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be $f(x,y) = (x^2 - y^2, 2xy)$. Prove that f is locally one-one but not one-one on $\mathbb{R}^2 \setminus (0,0)$ and discuss inverse function theorem at $(1,1)$.

6. Let $f \in \mathcal{R}[a,b]$ and $\{\phi_n\}$ be an orthonormal system in $C[a,b]$. Assume that $f \sim \sum c_n\phi_n$ and $s_n = \sum_{k=1}^n c_k\phi_n$. Then $\int_a^b |f - s_n|^2 \leq \int_a^b |f - t_n|^2$ where $t_n = \sum_{k=1}^n a_k\phi_n$ and constants a_k. Furthermore, the equality occurs only when $a_k = c_k$ for all k and $c_n \to 0$ as $n \to \infty$.

7. (a) Let f be a continuously differentiable map of an open set E of $\mathbb{R}^n$ into $\mathbb{R}^n$. If $f'(x)$ is invertible for every $x \in E$, prove that $f(U)$ is open whenever U is open in E (Marks: 3).

   (b) Let X be a compact metric space and g be a continuous function on $\mathbb{C}$. Prove that $\phi\colon C(X) \to C(X)$ defined by $\phi(f) = g \odot f$ is continuous (Marks: 3).